Lecture 13 on Oct. 28 2013

Today, we study the integration of an analytic function on closed curves. In what follows, R is a rectangle with length a and width b. Without loss of generality, we assume $a > b$. We use Δ to denote a disk. The first theorem is

Theorem 0.1. *If f is an analytic function in R, then*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0,$$

where ∂R is the boundary contour of R.

We can also weaken the assumption in Theorem 0.1 to get

Theorem 0.2. *If f is analytic on $R \setminus \{z_1, ..., z_n\}$ and moreover*

$$\lim_{z \to z_j} (z - z_j) f(z) = 0, \qquad \text{for all } j = 1, ..., n, \tag{0.1}$$

then it holds

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

We sketch the proof of Theorems 0.1-0.2 in the following. Reads should refer to the book of Ahlfors for more detailed arguments.

Proof of Theorem 0.1. Inductively if we have R_n a sub-rectangle of R, then we can bisect it into four identical rectangles, denoted by $R_{n,1}, R_{n,2}, R_{n,3}, R_{n,4}$, respectively. Clearly we have

$$\int_{\partial R_n} f(z)\,\mathrm{d}z = \int_{\partial R_{n,1}} f(z)\,\mathrm{d}z + \int_{\partial R_{n,2}} f(z)\,\mathrm{d}z + \int_{\partial R_{n,3}} f(z)\,\mathrm{d}z + \int_{\partial R_{n,4}} f(z)\,\mathrm{d}z.$$

Using triangle inequality, for some $i = 1, 2, 3, 4$, it must hold

$$\left| \int_{\partial R_{n,i}} f(z)\,\mathrm{d}z \right| \geq \frac{1}{4} \left| \int_{\partial R_n} f(z)\,\mathrm{d}z \right|.$$

Now we denote by R_{n+1} the $R_{n,i}$. Setting $R_1 = R$, we get a sequence of decreasing rectangles, say $\{R_n\}$, such that

$$\left| \int_{\partial R_{n+1}} f(z)\,\mathrm{d}z \right| \geq \frac{1}{4} \left| \int_{\partial R_n} f(z)\,\mathrm{d}z \right|, \qquad \text{for all } n \geq 1. \tag{0.2}$$

The above construction has four straightforward consequences. **1.** $R_n \longrightarrow z^*$ for some z^* in R; **2.** z^* must be in R_n for all n; **3.** for any z in R_n, the distance between z and z^* is bounded by the length of diagonal of R_n. More precisely

$$|z - z^*| \leq \text{length of diagonal of } R_n = \sqrt{\frac{a^2}{4^n} + \frac{b^2}{4^n}} < \frac{\sqrt{2}a}{2^n} < \frac{a}{2^{n-1}}; \tag{0.3}$$

4. the length of ∂R_n is bounded by

$$\text{length of } \partial R_n = \frac{a}{2^{n-1}} + \frac{b}{2^{n-1}} < \frac{a}{2^{n-2}}. \tag{0.4}$$

Since f is analytic at z^*, we have

$$\lim_{z \to z^*} \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| = 0.$$

Therefore for any $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ suitably small so that

$$|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \epsilon |z - z^*|, \qquad \text{for all } z \text{ with } |z - z^*| < \delta(\epsilon).$$

Noticing that R_n is shrinking to the point z^*, when n is large enough, any point w in R_n satisfies the condition $|w - z^*| < \delta(\epsilon)$. Therefore we know that for n large enough,

$$|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \epsilon |z - z^*|, \qquad \text{for all } z \text{ in } R_n.$$

Using this estimate, we know that

$$\left| \int_{\partial R_n} f(z)\, \mathrm{d}z \right| = \left| \int_{\partial R_n} f(z) - f(z^) - f'(z^*)(z - z^*)\, \mathrm{d}z \right| < \epsilon \int_{\partial R_n} |z - z^*|\, |\mathrm{d}z|.$$

Applying (0.3)-(0.4) to the right-hand side above, it follows that

$$\left| \int_{\partial R_n} f(z)\, \mathrm{d}z \right| < \frac{8a^2}{4^n}\, \epsilon.$$

By (0.2), one can easily show that

$$\left| \int_{\partial R_n} f(z)\, \mathrm{d}z \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f(z)\, \mathrm{d}z \right|.$$

Therefore the above two estimates show that

$$\left| \int_{\partial R} f(z)\, \mathrm{d}z \right| < 8a^2\, \epsilon.$$

Since ϵ is arbitrary, the proof is done. □

the proof of Theorem 0.2 is shown as follows.

Proof of Theorem 0.2. Without loss of generality, we assume f is analytic on $R \setminus \{z_0\}$. Letting R_n be a square centered at z_0 with dimension $1/2^n$. Clearly by Theorem 0.1, we have

$$\int_{\partial R} f(z)\, \mathrm{d}z = \int_{\partial R_n} f(z)\, \mathrm{d}z. \tag{0.5}$$

By the assumption in Theorem 0.2, we have

$$|z - z_0|\, |f(z)| < \epsilon, \qquad \text{provided that } |z - z_0| < \delta(\epsilon).$$

therefore when n is large enough, it follows

$$|z - z_0|\, |f(z)| < \epsilon, \qquad \text{for all } z \text{ in } R_n.$$

Applying the above estimate to (0.5), one can easily get

$$\left| \int_{\partial R} f(z)\, \mathrm{d}z \right| = \left| \int_{\partial R_n} f(z)\, \mathrm{d}z \right| \leq \epsilon \int_{\partial R_n} |z - z_0|^{-1} |\mathrm{d}z|.$$

Since z is on ∂R_n, $|z - z_0| \geq 1/2^{n+1}$. So the following estimate holds

$$\int_{\partial R_n} |z - z_0|^{-1} |\mathrm{d}z| \leq 2^{n+1} \frac{1}{2^{n-2}} = 8.$$

Using the above two estimate, we get

$$\left| \int_{\partial R} f(z)\, \mathrm{d}z \right| < 8\epsilon.$$

The proof is finished since ϵ is arbitrary. □

With Theorems 0.1-0.2, the following two results are trivial.

Theorem 0.3. *If f is analytic in Δ, then for all γ a closed curve in Ω, we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. Fixing z_0 in Δ, for any z in Δ, we can connect z_0 and z by vertical and horizontal segments. Define

$$F(z) = \int_\Gamma f(w)\,\mathrm{d}w,$$

where Γ connects z_0 and z. Meanwhile Γ is formed by vertical and horizontal segments. Using Theorem 0.1, $F(z)$ is independent of the choice of vertical and horizontal segments. Moreover, we also know that F is analytic and satisfies $f(z) = F'(z)$. Using the conclusion from Lecture 14, the proof is done. □

Same arguments can also be applied to show that

Theorem 0.4. *If f is analytic in $\Delta' = \Delta \setminus \{z_1, ..., z_n\}$ and*

$$\lim_{z\to z_j} (z - z_j) f(z) = 0, \qquad \text{, for all } j = 1, ..., n,$$

then we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0, \qquad \text{for all } \gamma \text{ a closed curve in } \Delta'.$$

One should notice that the γ in Theorem 0.4 can not pass the points in $\{z_1, ..., z_n\}$. Theorem 0.4 can be used to show the famous Cauchy integral formula. In fact, if f is analytic in Δ, then $F(z) = (f(z) - f(z_0))/(z - z_0)$ satisfies all assumptions in Theorem 0.4. Here z_0 is a point in Δ. Therefore if we have γ a closed curve in Δ, then

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z = 0,$$

provided that z_0 is not on γ. Rewrite the above equality, we get

$$f(z_0) \int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z = \int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{0.6}$$

In what follows, we try to understand the integral on the left-hand side of (0.6). Supposing that $z(t)$ is a parametrization of γ. t is running within the interval $[\alpha, \beta]$. Clearly we have $z(\alpha) = z(\beta)$ since γ is a closed curve. Letting

$$h(t) = \int_\alpha^t \frac{z'(s)}{z(s) - z_0}\,\mathrm{d}s, \qquad \text{for all } t \in [\alpha, \beta],$$

by fundamental theorem of calculus, one has

$$h'(t) = \frac{z'(t)}{z(t) - z_0}.$$

Defining

$$H(t) = e^{-h(t)}(z(t) - z_0),$$

then by product rule and chain rule, we have

$$H'(t) = e^{-h(t)}\left(z'(t) - h'(t)(z(t) - z_0)\right) = 0.$$

Thereofore $H(t)$ is a constant. it shows that

$$H(\beta) = e^{-h(\beta)}(z(\beta) - z_0) = H(\alpha) = z_\alpha - z_0.$$

Furthermore, we have $e^{h(\beta)} = 1$. that is $h(\beta) = 2k\pi i$, where k is some integer. $h(\beta)$ is the integral on the left-hand side of (0.6). Hence we know from (0.6) that

$$f(z_0) = \frac{1}{2\pi i k} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

One should notice that the integer k depends only on z_0 and the choice of closed curve γ. So in the following, we define this k to be the index of z_0 with respect to γ.

Definition 0.5. *Given z_0 and a closed curve γ, here z_0 is not on γ then we define*

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z.$$

$n(\gamma, z_0)$ is called the index of z_0 with respect to γ.